

# A Spectral Multigrid Method for the Stokes Problem in Streamfunction Formulation

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Suitable spectral multigrid components for the Stokes problem in streamfunction formulation are presented. We split the Stokes problem into a system of two equations with the Laplace operator. We discretize by means of a collocation (or pseudo spectral) method. Preconditioning with finite differences is employed. Numerical results for the spectral multigrid method are presented. © 1992 Academic Press, Inc.

## 1. INTRODUCTION

In the last few years a number of spectral techniques have been developed to solve the stationary Stokes equations in a two-dimensional domain  $\Omega = (-1, 1)^2$ , provided with Dirichlet boundary conditions. Most of them rely on the formulation of the problem where the unknowns are the velocity  $\mathbf{w} = (w_1, w_2)$  and the pressure  $p$ :

$$\begin{aligned} -\nu \Delta \mathbf{w} + \mathbf{grad} p &= \mathbf{g} & \text{in } \Omega, & (1.1) \\ \operatorname{div} \mathbf{w} &= 0 & \text{in } \Omega & (1.2) \end{aligned}$$

with homogeneous Dirichlet boundary conditions on  $\mathbf{w}$ . Here  $\nu > 0$  denotes the diffusion parameter and  $\mathbf{g} = (g_1, g_2)$  denotes a given vector-valued function defined in  $\Omega$ . In order to satisfy exactly Eq. (1.2) in the discrete problem, we introduce the streamfunction  $\psi$ . From Eq. (1.2) it follows that the velocity can be written as

$$\mathbf{w} = \mathbf{curl} \psi := \left( \frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x} \right).$$

Next, Eq. (1.1) is equivalent to the fourth-order equation:

$$\nu \Delta^2 \psi = \operatorname{curl} \mathbf{g} \quad \text{in } \Omega,$$

where  $\operatorname{curl} \mathbf{g} := (\partial/\partial y) g_1 - (\partial/\partial x) g_2$ . If we introduce a function  $f$  given by

$$f = \nu^{-1} \operatorname{curl} \mathbf{g} \quad \text{in } \Omega,$$

then (1.1) is equivalent to

$$\Delta^2 \psi = f \quad \text{in } \Omega. \tag{1.3}$$

The boundary conditions on the velocity induce boundary conditions on  $\psi$ :

$$\psi = \frac{\partial \psi}{\partial n} = 0 \quad \text{on } \partial \Omega, \tag{1.4}$$

where  $\partial/\partial n$  denotes the outer normal derivative on  $\partial \Omega$ .

For the purpose of developing a spectral multigrid (SMG) method we split Eq. (1.3) into an equivalent system of two equations with the Laplace operator. We further introduce a function  $\zeta$  defined in  $\bar{\Omega} = \Omega \cup \partial \Omega$ . Now the split system reads as

$$\begin{aligned} \Delta \psi - \zeta &= 0 & \text{in } \bar{\Omega}, & (1.5) \\ \Delta \zeta &= f & \text{in } \Omega \end{aligned}$$

with Dirichlet boundary conditions (1.4).

We discretize the system (1.5) by means of a collocation (or pseudo spectral) method. Convergence has been proven by Bernardi *et al.* [1] for a variational formulation of (1.5). In the one-dimensional case we proved convergence and derived optimal error estimates (see [10]). Vanel, Peyret, and Bontoux [17], Ehrenstein and Peyret [9] have developed  $\tau$  and collocation influence-matrix techniques for the streamfunction–vorticity formulation. Here we present a somewhat different collocation method which in numerical computations yields a somewhat higher spectral accuracy.

We further present a suitable finite difference (FD) discretization of the system (1.5). For an iterative method the FD operator is used for preconditioning. Without preconditioning the spectral operator has a condition number growing as  $O(N^4)$ , where  $N$  denotes the maximal degree of polynomials. By exact preconditioning we obtain an eigenvalue spectrum of  $[1, \pi^2/4]$  which is already known from

preconditioning of second-order model problems (see [5, 5.4]). Hence the reduction of the condition number is very impressive. A further reduction can be achieved by using finite element methods (see [5, 6, 8, 14]).

In the following we present effective SMG components. The transfer operators are standard (see [3, 12, 13, 18, 19]). For relaxation we employ a Richardson relaxation combined with defect correction. The defect correction is based on a FD approximation. Here we employ one step of an alternating line–Gauss–Seidel relaxation for approximating the solution of the FD problem. This type of relaxation turned out to be the most effective smoother for SMG methods (see [3, 12, 13]). An alternative is given by incomplete LU (ILU)-decomposition [12, 18, 19]. In earlier papers we found that this technique is much more expensive than line–Gauss–Seidel relaxation. The convergence factors are almost the same.

## 2. PSEUDO SPECTRAL DISCRETIZATION

In order to give the pseudo spectral discretization of (1.4), (1.5) we introduce the collocation nodes

$$(x_i, y_j) = \left( \cos \frac{i\pi}{N}, \cos \frac{j\pi}{N} \right), \quad i, j = 0, \dots, N, \quad (2.1)$$

in  $\bar{\Omega}$ . Clearly, the collocation nodes in  $\Omega$  are given by  $(x_i, y_j)$ ,  $i, j = 1, \dots, N-1$ .

The functions  $\psi$  and  $\zeta$  are spectrally approximated by polynomials  $\psi_{N+2} \in \mathbb{P}_{N+2}$ ,  $\zeta_N \in \mathbb{P}_N$ , where  $\mathbb{P}_M$ ,  $M \in \mathbb{N}$  denotes the subspace of polynomials of degree  $\leq M$  in the variables  $x$  and  $y$ . Furthermore,  $\psi_{N+2}$  is required to fulfill the Dirichlet boundary conditions. Hence the approximations  $\psi_{N+2}$ ,  $\zeta_N$  can be written as

$$\begin{aligned} \psi_{N+2}(x, y) &= (1-x^2)^2 (1-y^2)^2 \\ &\times \sum_{n,m=0}^{N-2} a_{n,m} T_n(x) T_m(y) \end{aligned} \quad (2.2)$$

and

$$\zeta_N(x, y) = \sum_{n,m=0}^N b_{n,m} T_n(x) T_m(y), \quad (2.3)$$

where  $T_k$  denotes the  $k$ th Chebyshev polynomial.

The pseudo spectral problem related to (1.4), (1.5) now reads as follows: Find  $\psi_{N+2}$  and  $\zeta_N$  such that

$$\begin{aligned} \Delta \psi_{N+2}(x_i, y_j) - \zeta_N(x_i, y_j) \\ = 0 \quad (i, j = 0, \dots, N), \end{aligned} \quad (2.4)$$

$$\begin{aligned} \Delta \zeta_N(x_i, y_j) \\ = f(x_i, y_j) \quad (i, j = 1, \dots, N-1). \end{aligned} \quad (2.5)$$

Hence Eqs. (2.4) and (2.5) yield  $M = (N+1)^2 + (N-1)^2 = 2(N^2+1)$  conditions of collocation for the  $M$  unknowns

$$a_{N-2} = (a_k)_{k=1, \dots, (N-1)^2}$$

and

$$b_N = (b_l)_{l=1, \dots, (N+1)^2},$$

where

$$\begin{aligned} a_k &= a_{n,m} \quad \text{for } k = m(N-1) + n + 1, \\ &\quad n, m = 0, \dots, N-2, \\ b_l &= b_{n,m} \quad \text{for } l = m(N+1) + n + 1, \\ &\quad n, m = 0, \dots, N. \end{aligned}$$

In order to write the system (2.4), (2.5) in matrix notation we further define the vectors

$$O_N \in \mathbb{R}^{(N+1)^2} \quad \text{zero vector,}$$

$$f_{N-2} = (f_k)_{k=1, \dots, (N-1)^2} \in \mathbb{R}^{(N-1)^2},$$

where

$$\begin{aligned} f_k &= f(x_i, y_j) \quad \text{for } k = (j-1)(N-1) + i, \\ &\quad i, j = 1, \dots, N-1 \end{aligned}$$

and matrices:

$$I \in \mathbb{R}^{(N+1)^2, (N+1)^2} \quad \text{unit matrix,}$$

$$O \in \mathbb{R}^{(N-1)^2, (N-1)^2} \quad \text{zero matrix,}$$

$$\bar{\Delta}_{\text{sp}} \in \mathbb{R}^{(N+1)^2, (N-1)^2}$$

spectral operator for  $\Delta$  applied to  $a_{N-2}$

and evaluated in all collocation points of  $\bar{\Omega}$ .

$$\Delta_{\text{sp}} \in \mathbb{R}^{(N-1)^2, (N+1)^2}$$

spectral operator for  $\Delta$  applied to  $b_N$

and evaluated in all collocation points of  $\Omega$ .

In matrix notation the spectral system (2.4), (2.5) can now be written as

$$\hat{L}_{\text{sp}} \begin{bmatrix} a_{N-2} \\ b_N \end{bmatrix} = \begin{bmatrix} \bar{\Delta}_{\text{sp}}, -I \\ O, \Delta_{\text{sp}} \end{bmatrix} \begin{bmatrix} a_{N-2} \\ b_N \end{bmatrix} = \begin{bmatrix} O_N \\ f_{N-2} \end{bmatrix}. \quad (2.6)$$

For an effective solution of (2.6) it is necessary to evaluate the spectral operator  $\hat{L}_{\text{sp}}$  in an effective way. Hence we have to look for a suitable evaluation of  $\Delta_{\text{sp}}$  and  $\bar{\Delta}_{\text{sp}}$ . The evalua-

tion of  $\Delta_{sp}$  can be efficiently accomplished by standard techniques introduced in previous papers by Zang *et al.* [18, 19] and Heinrichs [12, 13]. Here fast cosine transforms (based on (real) fast Fourier transforms (FFTs)) can be efficiently employed. The amount of computational work results in  $O(N^2 \ln N)$  arithmetic operations (see Temperton [15, 16]).

Therefore we only have to find an effective way of evaluating  $\bar{\Delta}_{sp}$ . Here we propose the following strategy. We first consider the one-dimensional evaluations of

$$(1 - y_j^2)^2 \sum_{m=0}^{N-2} b_m T_m(y_j), \quad b_m \in \mathbf{R} \quad (2.7)$$

and

$$\sum_{n=0}^{N-2} a_n [(1 - x^2)^2 T_n]''(x_i), \quad a_n \in \mathbf{R} \quad (2.8)$$

for  $i, j = 0, \dots, N$ . We observe that (2.7) can be efficiently evaluated in the following way: we set  $b_{N-1} = b_N = 0$ , apply a fast cosine transform to the sum up to  $N$ , and finally multiply the values in  $y_j$  by the term  $(1 - y_j^2)^2$ . This procedure requires a total amount of  $O(N \ln N)$  arithmetic operations. The effective evaluation of the sum (2.8) is more complicated. First we explicitly write the second derivative:

$$[(1 - x^2)^2 T_n(x)]'' = 4(3x^2 - 1) T_n(x) - 8x(1 - x^2) \times T_n'(x) + (1 - x^2)^2 T_n''(x). \quad (2.9)$$

Now we once more set  $a_{N-1} = a_N = 0$ . We further define matrices  $E$  and  $F$  which perform the multiplications by the functions  $x$ ,  $1 - x^2$  in Fourier space, i.e.,  $E \cong x$  and  $F \cong 1 - x^2$ . The entries of  $E$ ,  $F \in \mathbf{R}^{N+1, N+1}$  can be taken from the formulas written by Gottlieb and Orszag [11, Appendix (A.11), (A.12)]. They are explicitly given by  $E = (e_{k,l})_{k,l=1,\dots,N+1}$ ,  $F = (f_{k,l})_{k,l=1,\dots,N+1}$ , where

$$e_{k,l} = \begin{cases} \frac{1}{2}c_{k-1}, & l = k - 1 \\ \frac{1}{2}, & l = k + 1 \\ 0, & \text{else} \end{cases} \quad (2.10)$$

and

$$f_{k,l} = \begin{cases} 1 - \frac{1}{4}(c_k + c_{k-1}), & l = k \\ -\frac{1}{4}c_{k-2}, & l = k - 2 \\ -\frac{1}{4}, & l = k + 2 \\ 0, & \text{else} \end{cases} \quad (2.11)$$

with

$$c_k = \begin{cases} 1, & k \geq 2, \\ 2, & k = 1, \\ 0, & k < 1. \end{cases}$$

From these representations it becomes clear that  $E$  and  $F$  are sparse with 2 and 3 non-vanishing diagonals. We further denote by  $D \in \mathbf{R}^{N+1, N+1}$  the spectral matrix in Fourier space which corresponds to the first derivative (see [11, A.9]).

Here we write the corresponding formula explicitly;  $D = (d_{k,l})_{k,l=1,\dots,N+1}$  is given by

$$d_{k,l} = \begin{cases} \frac{2l}{c_k}, & l \geq k + 1, k + l \text{ odd}, \\ c_k, & \text{else.} \end{cases}$$

Since the non-vanishing entries of  $D$  in columns are identical (apart from the first row where the entries are divided by 2) the evaluation of  $D$  can be efficiently accomplished in  $O(N)$  arithmetic operations. Hence the total evaluation of (2.8) corresponds in Fourier space to the evaluation of the matrix  $D_{sp}$ , where

$$D_{sp} = 4(3E^2 - \hat{I}) - 8EFD + F^2D^2.$$

Here  $\hat{I} \in \mathbf{R}^{N+1, N+1}$  denotes the unity matrix. From the above considerations it becomes clear that the evaluation of  $D_{sp}$  requires a total amount of  $O(N)$  arithmetic operations. Since the resulting polynomial is a polynomial of degree  $N$ , the physical evaluation in (2.8) can be efficiently accomplished by a fast cosine transform. Altogether we conclude that the evaluation of  $\bar{\Delta}_{sp}$  only requires  $O(N^2 \ln N)$  arithmetic operations.

Since the iterative procedure (see Section 4) further requires FD preconditioning we also have to introduce the spectral operator in physical space. For this purpose we define vectors,

$$\psi_{N-2} = (\psi_k)_{k=1,\dots,(N-1)^2} \in \mathbf{R}^{(N-1)^2}$$

and

$$\zeta_N = (\zeta_l)_{l=1,\dots,(N+1)^2} \in \mathbf{R}^{(N+1)^2},$$

where

$$\psi_k = \psi_{N+2}(x_i, y_j) \quad \text{for } k = (j-1)(N-1) + i, \\ i, j = 1, \dots, N-1$$

and

$$\zeta_l = \zeta_N(x_i, y_j) \quad \text{for } l = j(N+1) + i + 1, \\ i, j = 0, \dots, N.$$

By an inverse transform,

$$T^{-1}: \begin{bmatrix} \psi_{N-2} \\ \zeta_N \end{bmatrix} \rightarrow \begin{bmatrix} a_{N-2} \\ b_N \end{bmatrix},$$

we map from physical space to Fourier space.

Now the spectral operator  $L_{\text{sp}}$  in physical space is given by

$$L_{\text{sp}} = \hat{L}_{\text{sp}} T^{-1}.$$

The inverse transform of  $\zeta_N$  to  $b_N$  can be accomplished by an inverse cosine transform. Hence it requires  $O(N^2 \ln N)$  arithmetic operations.

A procedure of determining  $a_{N-2}$  by means of FFTs is described in the following. For simplicity, we consider the one-dimensional case. The problem is that we want to determine the polynomial  $\tilde{\psi}_{N-2} \in \mathbf{P}_{N-2}$  by using the grid values  $\psi_{N+2}(x_i)$ ,  $i = 1, \dots, N-1$ , of the polynomial  $\psi_{N+2} \in \mathbf{P}_{N+2}$  given by

$$\psi_{N+2} = (1-x^2)^2 \tilde{\psi}_{N-2}, \quad \tilde{\psi}_{N-2} \in \mathbf{P}_{N-2}.$$

It is immediately seen that

$$\tilde{\psi}_N = \frac{\psi_{N+2}}{1-x^2} \in \mathbf{P}_N$$

and  $\tilde{\psi}_N(\pm 1) = 0$ . Hence we can calculate the coefficients of its Chebyshev series by using FFTs. Further, we have the identity

$$(1-x^2) \tilde{\psi}_{N-2} = \tilde{\psi}_N.$$

Since both polynomials vanish at the endpoints, it is enough to equate the Chebyshev coefficients of index  $\leq N-2$  on both sides. The Chebyshev coefficients on the left side can be expressed in terms of a three-term relation. The corresponding matrix is given by  $F = (f_{k,l})_{k,l=1,\dots,N-1}$  which is defined by (2.11).

In order to determine the Chebyshev coefficients of  $\tilde{\psi}_{N-2}$  we have to solve a system for  $F$  which can be accomplished in  $O(N)$  arithmetic operations. From numerical experiments we also found that this algorithm is robust with respect to roundoff errors.

In the two-dimensional case we have to solve a system for the matrix

$$F \otimes F = (Ff_{k,l})_{k,l=1,\dots,N-1} \in \mathbf{R}^{(N-1)^2, (N-1)^2},$$

where  $\otimes$  denotes the matrix tensor product. Solving a system with this matrix means solving a system for  $F$  twice. Hence the operational account for solving a system related to  $F \otimes F$  requires  $O(N^2)$  arithmetic operations. From these considerations it becomes clear that also  $a_{N-2}$  can be evaluated by FFTs which require a total amount of  $O(N^2 \ln N)$  arithmetic operations.

### 3. FINITE DIFFERENCE DISCRETIZATION

In order to present the FD discretization of the system (1.5) we first introduce the one-dimensional second derivative operator. Let  $D_{\text{FD}} \in \mathbf{R}^{N+1, N+1}$  denote the matrix which corresponds to the second derivative. In the collocation points  $x_i$ ,  $i = 1, \dots, N-1$ , we approximate by a three point star,

$$\psi''(x_i) \cong \frac{\beta}{s_i} \left[ \frac{1}{s_{i-1/2}}, -\frac{1}{s_{i-1/2}} - \frac{1}{s_{i+1/2}}, \frac{1}{s_{i+1/2}} \right] \psi,$$

where

$$\beta = 1/(2 \sin(\pi/2N) \sin(\pi/N)),$$

$$s_i = \sin(i\pi/N),$$

$$s_{i \pm 1/2} = \sin((i+1/2)\pi/N).$$

For a definition of the FD operator in boundary points we introduce an outer point. If we consider the boundary point  $x_0 = 1$  then we introduce  $x_{-1} = 2 - \cos(\pi/N)$  and after eliminating this point by means of the boundary condition we finally obtain in  $x = x_0$ ,

$$\psi''(x_0) \cong -\gamma\psi(x_0) + \gamma\psi(x_1),$$

where

$$\begin{aligned} \gamma &= \frac{2}{x_1 - x_{-1}} \left( \frac{1}{x_0 - x_{-1}} + \frac{1}{x_1 - x_0} \right) \\ &= \frac{2}{(1 - \cos(\pi/N))^2} \\ &= \frac{1}{2 \cdot \sin^4(\pi/2N)}, \end{aligned}$$

which behaves as  $O(N^4)$  for increasing  $N$ . In the same way we introduce around  $x_N$  an outer point  $x_{N+1} = -2 + \cos(\pi/N)$  and obtain

$$\psi''(x_N) \cong -\gamma\psi(x_N) + \gamma\psi(x_{N-1}).$$

Let us further define matrices  $\bar{D}_{\text{FD}} \in \mathbf{R}^{N+1, N-1}$ ,  $D_{\text{FD}}^0 \in \mathbf{R}^{N-1, N+1}$ .  $\bar{D}_{\text{FD}}$  corresponds to  $D_{\text{FD}}$ , where the first and last columns of  $D_{\text{FD}}$  are omitted (due to the homogeneous boundary conditions).  $D_{\text{FD}}^0$  corresponds to  $D_{\text{FD}}$ , where the first and last row are omitted (collocation only in points of  $\Omega$ ). Let us further denote by  $\bar{I} \in \mathbf{R}^{N+1, N-1}$ ,  $I^0 \in \mathbf{R}^{N-1, N+1}$  the matrices derived from the unit matrix  $\hat{I}$  in the same manner.

**TABLE I**  
 $\lambda_{\min}, \lambda_{\max}$  of  $L_{\text{FD}}^{-1}L_{\text{sp}}$ .

$N$	$\lambda_{\min}$	$\lambda_{\max}$
4	1.00	2.60
8	1.00	2.48
12	1.00	2.47

Finally by means of tensor products ( $\otimes$ ) we are able to write the discrete Laplace operators:

$$\begin{aligned}\bar{\Delta}_{\text{FD}} &= \bar{D}_{\text{FD}} \otimes \bar{I} + \bar{I} \otimes \bar{D}_{\text{FD}}, \\ \Delta_{\text{FD}} &= D_{\text{FD}}^0 \otimes I^0 + I^0 \otimes D_{\text{FD}}^0.\end{aligned}$$

Here  $\bar{\Delta}_{\text{FD}}$  yields an  $(N+1)^2 \times (N-1)^2$  matrix and  $\Delta_{\text{FD}}$  yields an  $(N-1)^2 \times (N+1)^2$  matrix. Now the FD system which corresponds to (1.5) can be written as

$$L_{\text{FD}} \begin{bmatrix} \psi_{N-2} \\ \zeta_N \end{bmatrix} = \begin{bmatrix} \bar{\Delta}_{\text{FD}}, -I \\ O, \Delta_{\text{FD}} \end{bmatrix} \begin{bmatrix} \psi_{N-2} \\ \zeta_N \end{bmatrix} = \begin{bmatrix} O_N \\ f_{N-2} \end{bmatrix}. \quad (3.1)$$

In the SMG method we use an iterative method, where  $L_{\text{FD}}$  is employed for preconditioning. For this reason it is interesting to study the exact preconditioning properties of  $L_{\text{FD}}^{-1}$ . In Table I we present the minimal and maximal eigenvalues of  $L_{\text{FD}}^{-1}L_{\text{sp}}$  for  $N=4, 8, 12$ . The minimal eigenvalue is always identically 1. The maximal eigenvalues approximate  $\pi^2/4 \doteq 2.467$  from above. Hence we obtain the same eigenvalue bounds as already observed for the Laplacian. Numerically we observe that some eigenvalues in the middle of the eigenvalue spectrum are complex with small imaginary part. We remark that a further improved preconditioning can be achieved by using bilinear finite elements (see [6, 8, 14]). But in the context with SMG methods (see [14]) it was found that, with respect to efficiency, finite element and FD preconditioning are comparable.

#### 4. SPECTRAL MULTIGRID METHOD

For a definition of the multigrid method we have to define the relaxation scheme and the grid transfer operators (interpolation and restriction). These multigrid components are used in a multigrid frame (see [2]). This means that we first use a relaxation sweep on the finest grid. Then the resulting residual is restricted to the next coarser grid. Here once more we approximate the coarse grid problem by a relaxation sweep. This procedure is continued until we arrive on the coarsest grid. This problem is solved exactly by a suitable direct solver (for instance, the Gauss method) or by a sufficient number of relaxation sweeps. By means of the

interpolation we afterwards correct the approximation up to the finest grid. In the following we describe the specific components of our SMG method.

##### 4.1. Relaxation

For relaxation we employ a Richardson (or Euler) scheme with defect correction (or preconditioning). If some approximations  $\tilde{\psi}_{N-2}, \tilde{\zeta}_N$  of the spectral solution are given, the calculation of the new approximations  $\hat{\psi}_{N-2}, \hat{\zeta}_N$  proceeds as follows:

###### 1. Defect computation.

$$\begin{bmatrix} \tilde{d}_N^1 \\ \tilde{d}_{N-2}^2 \end{bmatrix} = \begin{bmatrix} O_N \\ f_{N-2} \end{bmatrix} - L_{\text{sp}} \begin{bmatrix} \tilde{\psi}_{N-2} \\ \tilde{\zeta}_N \end{bmatrix}.$$

###### 2. Defect correction. Compute an approximation $[\hat{\psi}_{N-2}, \hat{\zeta}_N]$ to the exact solution of

$$L_{\text{FD}} \begin{bmatrix} \psi_{N-2} \\ \zeta_N \end{bmatrix} = \begin{bmatrix} \tilde{d}_N^1 \\ \tilde{d}_{N-2}^2 \end{bmatrix} \quad (4.1)$$

by one step of a line-Gauss Seidel relaxation.

###### 3. Richardson step.

$$\begin{bmatrix} \hat{\psi}_{N-2} \\ \hat{\zeta}_N \end{bmatrix} = \begin{bmatrix} \tilde{\psi}_{N-2} \\ \tilde{\zeta}_N \end{bmatrix} - \omega \begin{bmatrix} \tilde{\psi}_{N-2} \\ \tilde{\zeta}_N \end{bmatrix}$$

with a suitable relaxation parameter  $\omega$ .

The evaluation of the spectral residual (step 1) is the most expensive part of the relaxation procedure. The effective performance of this evaluation has already been described in Section 2. For the Richardson step (step 3) we have to define suitable relaxation parameters. In Section 3 we already observed an eigenvalue spectrum of  $[1, \pi^2/4]$  for the preconditioned spectral operator. Hence the optimal relaxation parameter for a stationary Richardson (SR) relaxation is given by  $\omega = 2/(1 + \pi^2/4) \doteq 0.5768$ . The resulting convergence factor becomes 0.4232. For the nonstationary Richardson (NSR) relaxation the relaxation parameter changes as the Richardson step changes. Here we recommend a sweep of three relaxations. The corresponding optimal relaxation parameters are given in [12]. The convergence factor results in 0.2797. We further remark that an adaptive parameter choice which, for instance, results from a minimal residual relaxation does not work, since—due to the Neumann boundary conditions—the symmetric part of the preconditioned spectral operator is indefinite (see [4]).

The most interesting part of the relaxation procedure is the defect correction, i.e., the iterative solution of the FD problem (4.1). From the previous considerations in [12, 13] it follows that line relaxation is necessary in order to obtain sufficient smoothing properties. Here we propose

an (alternating) line-Gauss-Seidel relaxation. For defect correction we employ one step of this relaxation procedure. Now we describe this type of relaxation in more detail.

First we introduce a five point star which yields an approximation of  $\Delta\psi$  in  $(x_i, y_j)$ ,  $i, j = 0, \dots, N$ :

$$(\Delta\psi)_{i,j} = \begin{bmatrix} 0, & f_{0,1}^{i,j}, & 0 \\ f_{-1,0}^{i,j}, & f_{0,0}^{i,j}, & f_{1,0}^{i,j} \\ 0, & f_{0,-1}^{i,j}, & 0 \end{bmatrix} \psi.$$

The entries of the five point star can be taken from the considerations in Section 3. In particular, it follows that:

$$f_{k,l}^{i,j} = 0 \quad \text{if } (x_{i+k}, y_{j+l}) \notin \bar{\Omega} \\ \text{for some } k, l \in \{-1, 0, 1\}.$$

Since the notation with stars requires boundary points we introduce a grid function  $\hat{\psi}_N$  which is identical to the homogeneous extension of  $\hat{\psi}_{N-2}$  to the boundary:

$$(\hat{\psi}_N)_{i,j} = \begin{cases} (\hat{\psi}_{N-2})_{i,j} & \text{for } i, j = 1, \dots, N-1, \\ 0 & \text{if } i \in \{0, N\} \text{ or } j \in \{0, N\}. \end{cases}$$

From (3.1) we derive in corner points:

$$(\hat{\zeta}_N)_{i,j} = -(\tilde{a}_N^1)_{i,j} \\ \text{for } (i, j) \in \{(0, 0), (0, N), (N, 0), (N, N)\}.$$

On the edges (without corners) we are able to express the boundary values of  $\hat{\zeta}_N$  by means of the values of  $\hat{\psi}_N$  in points next to the boundary:

$$(\hat{\zeta}_N)_{0,j} = f_{1,0}^{0,j}(\hat{\psi}_N)_{1,j} - (\tilde{a}_N^1)_{0,j}, \\ (\hat{\zeta}_N)_{N,j} = f_{-1,0}^{N,j}(\hat{\psi}_N)_{N-1,j} - (\tilde{a}_N^1)_{N,j} \quad (4.2)$$

for  $j = 1, \dots, N-1$  and

$$(\hat{\zeta}_N)_{i,0} = f_{0,1}^{i,0}(\hat{\psi}_N)_{i,1} - (\tilde{a}_N^1)_{i,0}, \\ (\hat{\zeta}_N)_{i,N} = f_{0,-1}^{i,N}(\hat{\psi}_N)_{i,N-1} - (\tilde{a}_N^1)_{i,N} \quad (4.3)$$

for  $i = 1, \dots, N-1$ .

By making use of these formulas it is possible to define the (alternating) line relaxation in points of  $\Omega$ , i.e., in  $(x_i, y_j)$ ,  $i, j = 1, \dots, N-1$ . Let us denote by  $\psi_N^0, \zeta_N^0$  the old approximations. We start with the zero grid approximation. Then the new approximations  $\hat{\psi}_N, \hat{\zeta}_N$  can be calculated in the following way:

One first relaxes along lines of constant  $x$  by solving for each  $j \in \{2, 4, \dots, N-2\}$ :

$$f_{-1,0}^{i,j}(\hat{\psi}_N)_{i-1,j} + f_{0,0}^{i,j}(\hat{\psi}_N)_{i,j} \\ + f_{1,0}^{i,j}(\hat{\psi}_N)_{i+1,j} - (\hat{\zeta}_N)_{i,j} \\ = (\tilde{a}_N^1)_{i,j} - f_{0,-1}^{i,j}(\hat{\psi}_N)_{i,j-1} \\ - f_{0,1}^{i,j}(\hat{\psi}_N)_{i,j+1}, \quad (4.4)$$

$$f_{-1,0}^{i,j}(\hat{\zeta}_N)_{i-1,j} + f_{0,0}^{i,j}(\hat{\zeta}_N)_{i,j} \\ + f_{1,0}^{i,j}(\hat{\zeta}_N)_{i+1,j} \\ = (\tilde{a}_N^2)_{i,j} - f_{0,-1}^{i,j}(\hat{\zeta}_N)_{i,j-1} \\ - f_{0,1}^{i,j}(\hat{\zeta}_N)_{i,j+1} \quad (i = 1, \dots, N-1). \quad (4.5)$$

Then the same sweep follows for the odd  $j$ , i.e.,  $j \in \{1, 3, \dots, N-1\}$ .

Afterwards an analogous sweep is performed along the lines of constant  $y$ . Using the relations (4.2), (4.3) the boundary values of  $\hat{\zeta}_N^0$  (resp.  $\hat{\zeta}_N$ ) are expressed by means of the values of  $\hat{\psi}_N^0$  (resp.  $\hat{\psi}_N$ ) in the points next to the boundary. After one complete relaxation sweep the boundary values of  $\hat{\zeta}_N$  can be updated by means of (4.2), (4.3). Hence we obtain systems of  $2(N-1)$  unknowns and equations.

The corresponding matrix can be split into four submatrices (see Fig. 1). The matrices on the diagonal are tridiagonal. The upper matrix is identical to the negative unit matrix. The lower matrix only contains nonvanishing entries in the first and last component of the diagonal.

Since a Gaussian elimination procedure applied to this type of matrix would fill up the matrix and destroy the sparse structure we recommend a renumbering of the grid points. If we, for instance, fix  $j \in \{1, \dots, N-1\}$  the above matrix is applied to the vector

$$[\psi_{1,j}, \psi_{2,j}, \dots, \psi_{N-1,j}, \zeta_{1,j}, \zeta_{2,j}, \dots, \zeta_{N-1,j}]^T.$$

Here we employ the abbreviation:  $\psi_{i,j} := (\hat{\psi}_N)_{i,j}$ ,  $\zeta_{i,j} := (\hat{\zeta}_N)_{i,j}$ .

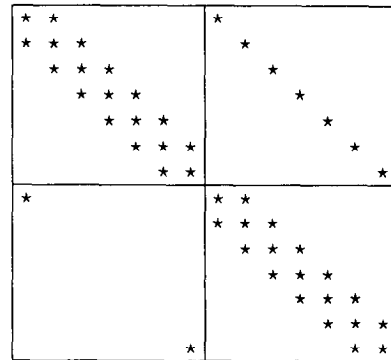


FIG. 1. Structure of submatrices for Gauss-Seidel line relaxation ( $N=8$ ).

TABLE II

 $\lambda_{\min}, \lambda_{\max}$  of  $L_{sp}$  Preconditioned by One Line-Gauss-Seidel Step

$N$	$\lambda_{\min}$	$\lambda_{\max}$
4	1.00	2.60
8	0.89	2.56
12	0.49	2.54
16	0.30	2.53
32	0.07	2.49

Now we renumber the components of this vector in the following way:

$$[\zeta_{1,j}, \psi_{1,j}, \zeta_{2,j}, \psi_{2,j}, \dots, \zeta_{N-1,j}, \psi_{N-1,j}]^T.$$

After a corresponding renumbering in the matrix we obtain a matrix with five non-vanishing diagonals (main diagonal, first and second diagonals, besides the main diagonal). For this type of matrix the Gaussian elimination procedure can be performed very efficiently. There is no fill up of the zero diagonals and hence the computational amount of work for solving the systems is proportional to  $N$ . One complete sweep of the line-Gauss-Seidel method requires a total amount of work of  $O(N^2)$  arithmetic operations. In Table II we present the minimal and maximal eigenvalues of the spectral operator preconditioned by one step of the above (alternating) line-Gauss-Seidel relaxation. The eigenvalues are numerically calculated by a vector iteration. The maximal eigenvalue once more approximates the value  $\pi^2/4$ , whereas the minimal eigenvalue tends to zero. For a multi-grid method the eigenvalue in the middle of the eigenvalue spectrum determines a lower bound. By numerical experiments we found this value to be about 1. This was also observed for the Laplace operator. Hence our choice of the relaxation parameter seems to be justified.

We also tried to replace  $(\zeta_N^0)_{i,0}, (\zeta_N^0)_{i,N}$  by  $(\zeta_N^0)_{i,0}, (\zeta_N)_{i,N}$  and  $(\zeta_N^0)_{0,j}, (\zeta_N^0)_{N,j}$  by  $(\zeta_N)_{0,j}, (\zeta_N)_{N,j}$ . This implies a change in the structure of the matrices for  $j=1, N-1$  and  $i=1, N-1$ . The zeros in the lower diagonals are replaced by non-zero components. The computational work for solving is nearly the same as before. The convergence factors are slightly improved. Hence in the following we resort to the previous treatment.

#### 4.2. Transfer Operators

For the grid transfers we employ the standard SMG restriction and interpolation operators (see [3, 12]). They consist of transforming the grid function on the finer grids, setting the higher Chebyshev coefficients to zero or appending additional zero coefficients as appropriate, and transforming back to the new coarser grid. In our context interpolation means that we add zero coefficients to the

Chebyshev coefficients  $a_{N-2}$  and  $b_N$ . The restriction operator is applied to the residuals  $\tilde{d}_N^1, \tilde{d}_{N-2}^2$  in a straightforward manner. Since the inverse transform also requires boundary values of  $\tilde{d}_{N-2}^2$  we extend this grid function by setting the boundary values to zero. This technique was successfully employed for the Poisson problem (see [12]).

## 5. NUMERICAL RESULTS

In the numerical computations we use a V-cycle (or W-cycle) with two ( $N=4, 8$ ), three ( $N=4, 8, 16$ ), four ( $N=4, 8, 16, 32$ ), and five ( $N=4, 8, 16, 32, 64$ ) grids. Fixed numbers  $n_d$  and  $n_u$  of relaxations on each grid in the downward and upward branches are employed. In order to measure the efficiency of our SMG method we calculate the computational amount of work. The standard work unit is the amount of work involved in one relaxation step on the finest grid. Hence we obtain, for instance, in the case of four grids:  $W = (1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64})(n_d + n_u) \cong 1.328125(n_d + n_u)$ . In order to estimate the convergence speed we compute the spectral radius  $\rho$  of the SMG operator by means of the power method. A convergence factor which is related to the work  $W$  is now defined by  $\rho_w = \rho^{1/W} \cdot \rho_w$  does not take the total work into account but it should be near the smoothing rate and provide an estimate of efficiency which is independent of both the computer and the programmer.

In Table III we present the spectral radii of the Richardson relaxation without using SMG. As it was expected the radii  $\rho$  approximate 1 for increasing  $N$ . This indicates a slow convergence speed of the Richardson iteration itself. In the Tables IV and V we present the numerical results for the SMG methods using a V-cycle (resp. W-cycle). For the stationary Richardson relaxation (SR) and nonstationary Richardson relaxation (NSR) we obtain the best results by choosing

$$\text{SR: } n_d = 2, n_u = 0,$$

$$\text{NSR: } n_d = 3, n_u = 0.$$

The numerical results show the highly improved convergence factors which are now about 0.5–0.6 also for increasing  $N$ . Further, it becomes clear that the W-cycle does not improve the convergence speed significantly. In

TABLE III

Results for the Richardson Iteration

$N$	$\rho$
8	0.5143
16	0.8145
32	0.9441
64	0.9688

**TABLE IV**  
Results for the SMG Method (V-Cycle)

Number of grids	Relaxation	$\rho$	$\rho_w$
2	SR	0.21	0.53
2	NSR	0.09	0.52
3	SR	0.23	0.57
3	NSR	0.08	0.52
4	SR	0.28	0.61
4	NSR	0.08	0.53
5	SR	0.35	0.67
5	NSR	0.10	0.56

comparison with the numerical results for the Poisson equation (see [12]) the convergence factors are somewhat worse (about 0.13 in  $\rho_w$  for the V-cycle with four grids). This small deterioration of the convergence factors is probably caused by the necessity to enforce the two types of boundary conditions on the streamfunction  $\psi$ . This does not fit to a second-order operator and perturbs the effect of smoothing. At the moment it is not clear how to avoid this perturbation in convergence speed.

Finally, we give some numerical results for two examples. Here we first consider problem (1.3), where the exact solution is given by

$$\psi(x, y) = \sin^2(2\pi x) \sin^2(2\pi y). \quad (5.1)$$

Second, we consider problem (1.3), where

$$f \equiv 1. \quad (5.2)$$

Both examples are associated with the boundary conditions (1.4). In example (5.1) we have a smooth solution, whereas in example (5.2) we have a solution which has a mild singularity in the four corners. The singularity is due to the fact that the right-hand side  $f \equiv 1$  and the boundary conditions are not compatible. The boundary conditions enforce  $\Delta^2\psi = 0$  in the corners, whereas we require  $\Delta^2\psi = 1$  in the domain  $\Omega$ . A similar example was considered in [12] for the

**TABLE V**  
Results for the SMG Method (W-cycle)

Number of grids	Relaxation	$\rho$	$\rho_w$
2	SR	0.20	0.52
2	NSR	0.09	0.52
3	SR	0.19	0.53
3	NSR	0.10	0.55
4	SR	0.28	0.61
4	NSR	0.07	0.51
5	SR	0.34	0.67
5	NSR	0.09	0.54

Poisson equation. Clearly, the solution is now more regular than for the second-order problem. By using standard regularity arguments it can be shown that  $\psi \in H^4(\Omega)$ , where  $H^4(\Omega)$  denotes the Sobolev space which contains the four times (weakly) differentiable functions.

By executing a sufficient number of multigrid steps we compute the absolute discretization errors:

$$E2 = \|\psi_{N+2} - \psi\|_2 \quad \text{and} \quad EM = \|\psi_{N+2} - \psi\|_{\max}$$

in the discrete  $l_2$ -norm  $\|\cdot\|_2$  and the maximum norm  $\|\cdot\|_{\max}$  given by

$$\|\xi\|_2 = \frac{1}{N} \sqrt{\sum_{i,j=0}^N \xi^2(x_i, y_j)},$$

$$\|\xi\|_{\max} = \text{Max}\{|\xi(x_i, y_j)| : i, j = 0, \dots, N\}.$$

Then we count the number  $IT$  of V-cycles needed in order to achieve an accuracy of

$$\|\psi_{N+2}^{(IT)} - \psi_{N+2}\|_2 < E2$$

for the  $IT$ th approximation  $\psi_{N+2}^{(IT)}$ . Here we start with the zero grid function. Now the convergence rate can be measured by the mean value:

$$\bar{\rho} = \left( \frac{\|\psi_{N+2}^{(IT)} - \psi_{N+2}\|_2}{\|\psi_{N+2}\|_2} \right)^{1/IT}.$$

The convergence factor per work unit is given by  $\bar{\rho}_w = \bar{\rho}^{1/W}$ . The numerical results are presented in the Tables VI and VII. They demonstrate the high accuracy of our spectral method which is exponential in the case of the smooth solution (5.1). For example (5.2), the spectral method does not converge exponentially fast due to the singularities in the corners. Nine V-cycles are always enough to reach the truncation error. The convergence rate is always below  $\frac{1}{2}$ . These numerical results are quite similar to the results for the Poisson equation in [12]. Hence we found a SMG method for the Stokes problem in streamfunction-vorticity formulation which similarly works as well as for the Poisson equation.

**TABLE VI**  
Results for Example (5.1)

$N$	$EM$	$E2$	$IT$	$\bar{\rho}_w$
8	$2.56 \times 10^1$	$9.36 \times 10^0$	1	0.37
16	$5.89 \times 10^{-3}$	$1.35 \times 10^{-3}$	2	0.44
32	$2.17 \times 10^{-13}$	$5.27 \times 10^{-14}$	9	0.44



**TABLE VII**  
Results for Example (5.2)

$N$	$EM$	$E2$	$IT$	$\bar{\rho}_w$
8	$1.55 \times 10^{-6}$	$4.35 \times 10^{-7}$	6	0.44
16	$4.69 \times 10^{-9}$	$1.19 \times 10^{-9}$	8	0.47
32	$7.34 \times 10^{-11}$	$5.31 \times 10^{-12}$	8	0.48

## 6. CONCLUSIONS

We presented an efficient spectral multigrid method for the Stokes problem in streamfunction–vorticity formulation. Here we have two boundary conditions for the streamfunction and none for the vorticity. Hence we introduce a spectral collocation method where the streamfunction is approximated by polynomials in  $\mathbf{P}_{N+2}$ , whereas the vorticity is approximated by polynomials in  $\mathbf{P}_N$ . The corresponding spectral operator can be efficiently preconditioned by a suitable finite difference scheme. Hence for relaxation in SMG we propose a preconditioned Richardson relaxation. The components of the SMG method are standard. From the numerical results (convergence factors) it can be seen that our SMG method for the Stokes equation yields a similar efficiency as already observed for the Poisson equation.

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